Complex Scaled Infinite Elements for Wave Equations in Heterogeneous Open Systems

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Abstract

The technique of *complex scaling* is a popular way to deal with the *wave equation* on unbounded domains. It is based on a complex coordinate stretching in the time harmonic regime. In our work we consider settings, where the usual cartesian or radial scalings are not applicable due to inhomogeneous exterior domains (e.g. open waveguides in non-axial directions). We apply a scaling in *normal* direction. Moreover we use *infinite elements* to discretize the complex scaled equation instead of truncating the domain to benefit from superior approximation properties and omit an additional truncation error. We present numerical experiments to illustrate our results.

Keywords: wave equation, infinite elements, complex scaling

1 Introduction

We consider numerically solving the wave equation

$$c(\mathbf{x})^2\,\Delta_{\mathbf{x}} p(t,\mathbf{x}) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} p(t,\mathbf{x}) \qquad (1)$$

on $\Omega := \mathbb{R}^2$. We are interested in settings similar to the one sketched in Figure 1, where the wave speed c is constant inside and outside of a set of open waveguides respectively. More generally we assume that there exist $\Omega_{\text{int}}, \Gamma, \Omega_{\text{ext}} \subset \mathbb{R}^2$, such that $\Omega = \Omega_{\text{int}} \dot\cup \Gamma \dot\cup \Omega_{\text{ext}}$, where Ω_{int} is open, bounded, and convex, $\Gamma = \partial\Omega_{\text{int}}$ is smooth with outer normal $\mathbf{n}$,

$$\Omega_{\text{ext}} = \left\{\hat{\mathbf{x}} + \xi\mathbf{n}(\hat{\mathbf{x}}) : \xi \in \mathbb{R}_{>0}, \hat{\mathbf{x}} \in \Gamma\right\}, \qquad (2)$$

such that the coordinates $\xi(\mathbf{x}), \hat{\mathbf{x}}(\mathbf{x})$ are unique for each $\mathbf{x} \in \Omega_{\text{ext}}$ and

$$c|_{\Omega_{\text{ext}}}(\mathbf{x}) = \tilde{c}(\xi(\mathbf{x}))\,\hat{c}(\hat{\mathbf{x}}(\mathbf{x}))\,.$$

Since we allow inhomogeneities which are neither radial nor parallel to the axes, the frequently used cartesian or radial complex scalings are not applicable in this case.

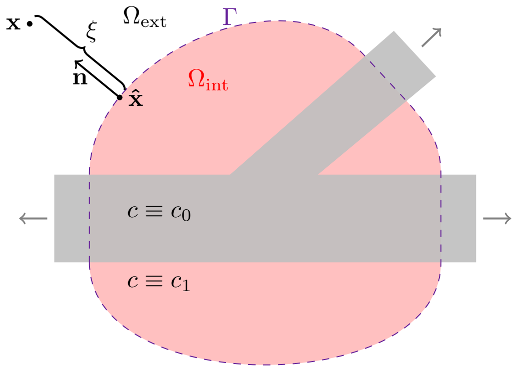

Figure 1: Example domain, where cartesian or spherical scalings would fail: Open waveguide with junction.

2 Absorbing layers for the wave equation

To construct absorbing boundary layers for the wave equation we follow the ideas presented for example in [4]. Note that the following steps are merely theoretical prerequisites to our method. The numerical method itself consists of discretizing the resulting system of equations. First, we apply a Fourier transformation. Then the technique of complex scaling is applied to the resulting *Helmholtz equation*. This technique relies on a complex coordinate stretching $\gamma : \Omega_{\text{ext}} \to \mathbb{C}^2$, which is chosen such that the outgoing solutions of the Helmholtz equation on the complex scaled domain $\gamma(\Omega_{\text{ext}})$ are exponentially decaying. We use normal scalings (cf. [1]), meaning that the complex deformation $\gamma(\mathbf{x}(\hat{\mathbf{x}}, \xi)) := x(\hat{\mathbf{x}}, s(\xi))$ for a scalar scaling function $s : \mathbb{R}_{>0} \to \mathbb{C}$, is only applied to the normal coordinate ξ. The scaling function s is of the form

$$s_\omega(\xi) = \frac{q_1(\mathrm{i}\omega)}{q_2(\mathrm{i}\omega)}\xi \qquad (3)$$

with complex polynomials q_1, q_2. While the dependency on the frequency is advantageous for resonance problems (cf. [3]) due to better approximation properties, it is essential for time domain. The wrong treatment of a subset of the present frequencies can lead to exponentially

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growing and therefore unstable solutions in time. Different choices of (3) are suitable for different problem settings.

The resulting complex scaled equation is subsequently transformed back to time domain by applying the according inverse Fourier transformation. Since powers of $-\mathrm{i}\omega$ are transformed to time derivatives, we introduce a suitable set of additional unknowns to end up with a second order system in time again.

3 Discretization in space

To discretize the system obtained in the previous section in space, usually the finite element method is applied to a truncated exterior domain which introduces an additional error. Differing from this approach we use infinite elements which are based on Hardy space infinite elements (cf. [2]) instead. To this end, we use the exterior coordinates $\xi, \hat{\mathbf{x}}$ and a tensor product ansatz space (cf. Figure 2). The first part

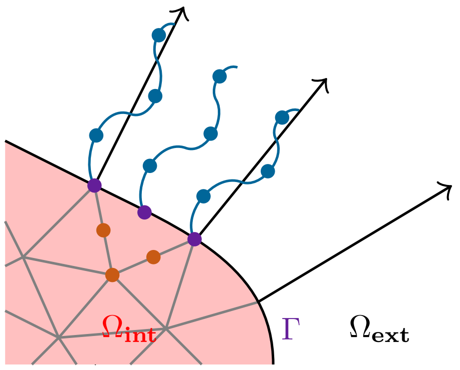

Figure 2: Sketch of the tensor product basis and some of the degrees of freedom for triangular finite elements of order 2 and infinite elements of order 3.

of this space is composed of boundary functions, which are the traces of the basis functions of the interior discretization. The second part consists of basis functions in the normal coordinate

$$\phi_j(\xi) = \exp(-\xi)\, p_j(\xi)\,,$$

for certain polynomials p_j of degree j (cf. Figure 3). These polynomials are closely linked to the *Laguerre polynomials* which form a complete orthogonal system for a weighted L^2-space on $\mathbb{R}_{>0}$.

Desirable properties of the basis functions ϕ_j include that they result in sparse, well-conditioned discretization matrices. Moreover it can be shown that they approximate the normal components of the solution super-algebraically with respect to the number of basis functions. They are simple to evaluate and can be integrated numerically. Coupling the interior and exterior problem works in a straightforward way.

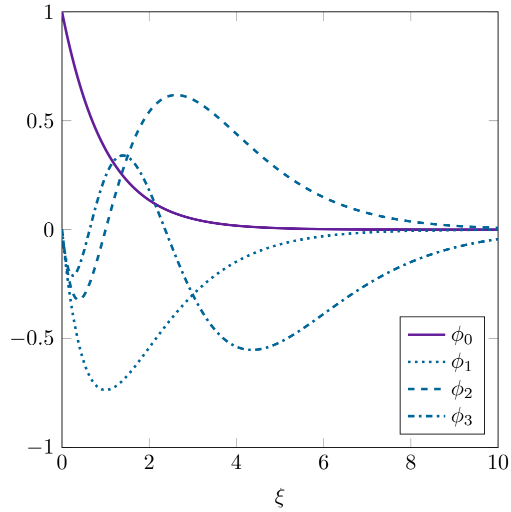

Figure 3: The first few basis functions ϕ_j. The basis function ϕ_0 couples with the interior and corresponds to the degrees of freedom on the interface Γ in Figure 2, while the remaining basis functions live purely in the exterior.

4 Time integration

After spacial discretization the resulting semi-discrete system in time is discretized using implicit time-stepping methods. A possible extension of our method would be the use of explicit time-stepping schemes to improve computational efficiency. To this end, a discontinuous Galerkin approach for the interior and interface basis functions can be used.

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